

Available online at www.sciencedirect.com**SciVerse ScienceDirect**

Advances in Mathematics 231 (2012) 2194–2223

**ADVANCES IN
Mathematics**www.elsevier.com/locate/aim

The Q -curvature on a 4-dimensional Riemannian manifold (M, g) with $\int_M Q dV_g = 8\pi^2$

Jiayu Li^{a,b,*}, Yuxiang Li^c, Pan Liu^d^a School of Mathematical Science, University of Science and Technology of China, Hefei 230026, PR China^b Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100080, PR China^c Department of Mathematical Sciences, Tsinghua University, Beijing 100084, PR China^d Department of Mathematics, East China Normal University, 3663, Zhong Shan North Rd, Shanghai 200062, PR China

Received 4 March 2011; accepted 5 June 2012

Available online 14 August 2012

Communicated by Gang Tian

Abstract

We deal with the Q -curvature problem on a 4-dimensional compact Riemannian manifold (M, g) with $\int_M Q_g dV_g = 8\pi^2$ and positive Paneitz operator P_g . Let \tilde{Q} be a positive smooth function. The question we consider is, when can we find a metric \tilde{g} which is conformal to g , such that \tilde{Q} is just the Q -curvature of \tilde{g} . A sufficient condition to this question is given in this paper.

© 2012 Elsevier Inc. All rights reserved.

Keywords: Q -curvature 4-dimensional Riemannian manifold

1. Introduction

One of the most important problems in conformal geometry is the construction of conformal metrics for which a certain curvature quantity equals a prescribed function, e.g. a constant. In two dimensions, the problem of prescribed Gaussian curvature asks the following: given a smooth function K on (M, g_0) , can we find a metric g conformal to g_0 such that K is the Gaussian

* Corresponding author at: School of Mathematical Science, University of Science and Technology of China, Hefei 230026, PR China.

E-mail addresses: jiayuli@ustc.edu.cn (J. Li), yxli@math.tsinghua.edu.cn (Y. Li), pliu@math.ecnu.edu.cn (P. Liu).

curvature of the new metric g ? If we let $g = e^{2u}g_0$ for some $u \in C^\infty(M)$, then the problem is equivalent to solving the nonlinear elliptic equation:

$$\Delta u + K e^{2u} - K_0 = 0, \quad (1.1)$$

where Δ denotes the Beltrami–Laplacian of (M, g_0) and K_0 is the Gaussian curvature of g_0 .

In dimension four, there is an analogous formulation of Eq. (1.1). Let (M, g) be a compact Riemannian four manifold, and let Ric and R denote respectively the Ricci tensor and the scalar curvature of g . A natural conformal invariant in dimension four is

$$Q = Q_g = -\frac{1}{12}(\Delta R - R^2 + 3|Ric|^2).$$

Note that, under a conformal change of the metric

$$\tilde{g} = e^{2u}g,$$

the quantity Q transforms according to

$$2Q_{\tilde{g}} = e^{-4u}(Pu + 2Q_g), \quad (1.2)$$

where $P = P_g$ denotes the Paneitz operator with respect to g , introduced in [18]. The operator P_g acts on a smooth function u on M via

$$P_g(u) = \Delta_g^2 u + \operatorname{div} \left(\frac{2}{3} R_g - 2Ric_g \right) du,$$

which plays a similar role as the Laplace operator in dimension two. Note that the Paneitz operator is conformally invariant in the sense that

$$P_{\tilde{g}} = e^{-4u} P_g$$

for any conformal metric $\tilde{g} = e^{2u}g$.

It follows from (1.2) that the expression $k = k_g := \int_M Q dV_g$ is conformally invariant. A natural problem to propose is to prescribe the Q -curvature: that is, to ask whether on a given four-manifold (M, g) there exists a conformal metric $\tilde{g} := e^{2u}g$ for which the Q -curvature of \tilde{g} equals the prescribed function \tilde{Q} . This is related to solving the following equation

$$P_g u + 2Q_g = 2\tilde{Q}e^{4u}. \quad (1.3)$$

This equation is the Euler–Langrange equation of the functional

$$II_g(u) = \int_M u P_g u dV_g + 4 \int_M Q_g u dV_g - \left(\int_M Q_g dV_g \right) \log \int_M \tilde{Q} e^{4u} dV_g. \quad (1.4)$$

A partial affirmative answer to the problem (1.3) in the case where \tilde{Q} equals some constant is given by Chang–Yang [3] provided the Paneitz operator is weakly positive and the integral k is less than $8\pi^2$. In view of the result of Gursky [9] the former hypothesis is satisfied whenever $k > 0$ and provided (M, g) is of positive Yamabe type. The result of Chang–Yang has been extended recently by Djadli–Malchiodi [7] to the case in which P_g has no kernel and k is not a positive integer multiple of $8\pi^2$.

In the critical case, when $k = 8\pi^2$, the study of Eq. (1.3) becomes rather delicate. In this case, the functional II_g fails to satisfy standard compactness conditions like the Palais–Smale

condition, and generally blow-up may occur. Note that when $(M, g) = (S^4, g_c)$, Eq. (1.3) is reduced to the following one

$$P_g u + 6 = 2\tilde{Q}e^{4u}. \quad (1.5)$$

This is the analogue of the well-known Nirenberg's problem. We should mention that, the blow-up phenomena for the Paneitz operator and other 4-th order elliptic equations have been deeply studied by Druet–Robert [8] and Weinstein–Zhang [21]. For other recent results, one can refer to [1,2,5,4,15,19,20,16]. We remark that, similar to Nirenberg's problem, there are some obstructions for the existence of the solution to Eq. (1.5) in the standard four-sphere case. The Gauss–Bonnet–Chern formula implies that there could not be a solution if $\tilde{Q} \leq 0$. On the other hand, one has the identities of Kazdan–Warner type to this equation.

The main goal of this paper is to study Eq. (1.3) with critical value $k = 8\pi^2$ and positive \tilde{Q} . We shall pursue a variational approach which was used in [6]. Let (M, g) be any closed four dimensional Riemannian manifold with positive P_g , i.e., $\int_M u P_g u dV_g \geq 0$ and $\ker P_g = \{\text{constants}\}$. Then we have

$$\int_M u P_g u dV_g \geq \lambda \int_M |\nabla_g u|^2 dV_g, \quad \text{when } \int_M u dV_g = 0 \quad (1.6)$$

for some positive λ and the following improved Adams–Fontana inequality [3]:

$$\log \int_M e^{4u} dV_g \leq \frac{1}{8\pi^2} \int_M u P_g u dV_g + \frac{1}{2\pi^2} \int_M u dV_g + C, \quad \forall u \in W^{2,2}(M). \quad (1.7)$$

We consider (for any small $\epsilon > 0$)

$$II_\epsilon(u) = \int_M \langle u, u \rangle dV_g + 4 \left(1 - \frac{\epsilon}{8\pi^2}\right) \int_M Q_g u dV_g - (8\pi^2 - \epsilon) \log \int_M \tilde{Q} e^{4u} dV_g,$$

where we denote

$$\langle u, v \rangle = \Delta_g u \Delta_g v + \left(\frac{2}{3} R_g(\nabla u, \nabla v) - 2\text{Ric}_g(\nabla u, \nabla v)\right).$$

By using the inequality (1.7), it is not so difficult to prove that

$$\inf II_\epsilon(u) > -\infty, \quad \forall \epsilon > 0, \text{ and moreover, } II_\epsilon \text{ has a minimum point } u_\epsilon.$$

For this minimizing sequence u_ϵ , two possibilities may occur: let $m_\epsilon = u_\epsilon(x_\epsilon) = \max_{x \in M} u_\epsilon(x)$,

- (1) $\sup_\epsilon m_\epsilon < +\infty$, then, by passing to a subsequence, $\{u_\epsilon\}$ converges to some u_0 as $\epsilon \rightarrow 0$, and u_0 minimizes II ;
- (2) $m_\epsilon \rightarrow +\infty$, as $\epsilon \rightarrow 0$; We say, in this case, the u_ϵ blows up.

One of the main concern is to prove that, if the second case happens, then we find an explicit bound for the II_ϵ . More precisely, we have

$$\inf_{u \in W^{2,2}(M)} II(u) \geq A_g(\tilde{Q}, p), \quad (1.8)$$

where

$$\begin{aligned} \Lambda_g(\tilde{Q}, p) = & -16\pi^2 \log \frac{\sqrt{3\tilde{Q}(p)}}{12} - 8\pi^2 \log 8\pi^2 - 16\pi^2 S_0(p) \\ & + 2 \int_M Q G_p dV_g + (8/3 - 16)\pi^2, \end{aligned}$$

p is the bubble point, and $S_0(p)$ is the constant term of the Green function at point p (see Appendix).

On the other hand, if we can construct some test function sequence ϕ_ϵ , s.t.

$$II(\phi_\epsilon) < \Lambda_g(\tilde{Q}, p),$$

we see that the blow-up does not happen. Therefore, we can get some sufficient condition under which (1.3) has a solution.

One of our main theorems in this paper is as follows.

Theorem 1.1. *Let (M, g) be a closed Riemannian manifold of dimension four, with $k = 8\pi^2$. Suppose P_g is positive and $\tilde{Q} > 0$. If $\inf_{u \in W^{2,2}(M)} II(u)$ is not attained, i.e. Eq. (1.3) has no minimal solution, then*

$$\inf_{u \in W^{2,2}(M)} II(u) = \inf_{p \in M} \Lambda_g(\tilde{Q}, p). \quad (1.9)$$

Now let p' be a point s.t.

$$\Lambda_g(\tilde{Q}, p') = \inf_{x \in M} \Lambda_g(\tilde{Q}, x),$$

we will prove that p' is in fact determined by the conformal class $[g]$ of (M, g) .

Another main result in this paper is the existence theorem of Eq. (1.3).

Theorem 1.2. *Let (M, g) be a closed Riemannian manifold of dimension four, with $k = 8\pi^2$. Suppose P_g is positive. Let \tilde{Q} be a positive smooth function on M . Assume that $\Lambda_g(\tilde{Q}, x)$ achieves its minimum at the point p' . If*

$$\tilde{Q}(p') \left(\Delta_g S(p') + 4|\nabla_g S(p')|^2 - \frac{R(p')}{18} \right) + \left[(2\nabla_g S \nabla_g \tilde{Q})(p') + \frac{1}{4} \Delta_g \tilde{Q}(p') \right] > 0,$$

then Eq. (1.3) has a minimal solution.

Corollary 1.3. *Under the assumption as in Theorem 1.2, if*

$$\Delta_g S(p') + 4|\nabla_g S(p')|^2 - \frac{R(p')}{18} > 0,$$

then M has a constant Q -curvature up to conformal transformations.

It is interesting to note that, in the four-dimensional case, the method in [6] cannot be directly used. Since Eq. (1.3) does not satisfy the Maximum Principle, the method used in [6] does not work here to calculate

$$\int_{B_\delta \setminus B_{Lr_\epsilon}(x_\epsilon)} |\Delta_g u_\epsilon|^2 dV_g. \quad (1.10)$$

We will apply the capacity to get the lower bound of (1.10). The usefulness of capacity in similar problems was first discovered by the second author, and has been used in [11,12].

2. Preliminary estimate

In this section we collect some useful preliminary facts and then derive some estimates for the solutions. We start with the following lemma.

Lemma 2.1. *For any $\epsilon > 0$, II_ϵ has a minimum point.*

Proof. By using the inequality (1.7), it is easy to see that, when $\int_M u dV_g = 0$, we have

$$\begin{aligned} II_\epsilon(u) &= \int_M u P_g u dV_g + 4 \left(1 - \frac{\epsilon}{8\pi^2}\right) \int_M Q u dV_g - (8\pi^2 - \epsilon) \log \int_M \tilde{Q} e^{4u} dV_g \\ &\geq C + \frac{\epsilon}{8\pi^2} \int_M u P_g u dV_g + 4 \left(1 - \frac{\epsilon}{8\pi^2}\right) \int_M Q u dV_g \\ &\geq C + \lambda \frac{\epsilon}{8\pi^2} \int_M |\nabla_g u|^2 dV_g + 4 \left(1 - \frac{\epsilon}{8\pi^2}\right) \int_M Q u dV_g. \end{aligned}$$

For any $\epsilon_1 > 0$, we have

$$\int_M Q u dV_g \leq \epsilon_1 \int_M |u|^2 + C_\epsilon \leq \lambda_0 \epsilon_1 \int_M |\nabla u|^2 dV_g + C_\epsilon,$$

where λ_0 is the first eigenvalue of Δ . Then,

$$\int_M |\nabla_g u|^2 dV_g \leq C(\epsilon) II_\epsilon(u) + C \quad (2.1)$$

and then

$$\int_M |\Delta_g u|^2 dV_g \leq \frac{8\pi}{\epsilon} II_\epsilon(u) + C. \quad (2.2)$$

Let $u_k = u_{\epsilon,k}$ be a minimizing sequence of II_ϵ , i.e.

$$II_\epsilon(u_k) \rightarrow \inf II_\epsilon(u) = A,$$

which, together with the above inequality, implies that

$$\int_M |\Delta_g u_k|^2 dV_g \leq C,$$

for some constant C which may depend on ϵ . Therefore, by passing to a subsequence, we have $u_k \rightharpoonup u_\epsilon$ and

$$\int_M |\Delta_g u_k|^2 dV_g \rightarrow B.$$

Since the functional II_ϵ is invariant under a translation by a constant, we may assume that $\int_M u_k dV_g = 0$, then by (1.7), we can see that $e^{4u_k} \in L^p$ for any $p > 0$.

Set

$$II_\epsilon(u_k) := \int_M |\Delta_g u_k|^2 dV_g + \int_M F(u_k) dV_g,$$

then we have,

$$\lim_{k \rightarrow +\infty} \int_M F(u_k) dV_g = A - B, \quad \text{and}$$

$$\lim_{k \rightarrow +\infty, m \rightarrow +\infty} \int_M F\left(\frac{u_k + u_m}{2}\right) dV_g = A - B.$$

Since $II_\epsilon\left(\frac{u_k + u_m}{2}\right) \geq A$, we have

$$\frac{1}{4} \int_M (|\Delta_g u_k|^2 + |\Delta_g u_m|^2) dV_g + \frac{1}{2} \int_M \Delta_g u_k \Delta_g u_m dV_g \geq B.$$

Hence

$$\lim_{k \rightarrow +\infty, m \rightarrow +\infty} \int_M \Delta_g u_k \Delta_g u_m dV_g \geq B.$$

Then

$$\begin{aligned} & \lim_{k \rightarrow +\infty, m \rightarrow +\infty} \int_M |\Delta_g(u_k - u_m)|^2 dV_g \\ &= \lim_{k \rightarrow +\infty, m \rightarrow +\infty} \left(\int_M |\Delta_g u_k|^2 dV_g + \int_M |\Delta_g u_m|^2 dV_g - 2 \int_M \Delta_g u_k \Delta_g u_m dV_g \right) \\ &\leq 0. \end{aligned}$$

Therefore, $\{u_k\}$ is a Cauchy sequence in $W^{2,2}(M)$. \square

Lemma 2.2. *We have*

$$\liminf_{\epsilon \rightarrow 0} II_\epsilon = \inf II.$$

Proof. Obviously,

$$\begin{aligned} II_\epsilon(u) &= \int_M u P_g u dV_g + 4 \left(1 - \frac{\epsilon}{8\pi^2}\right) \int_M Q u dV_g - (8\pi^2 - \epsilon) \log \int_M \tilde{Q} e^{4u} dV_g \\ &= \int_M u P_g u dV_g + 4 \int_M Q u dV_g - 8\pi^2 \log \int_M \tilde{Q} e^{4u} dV_g \\ &\quad - \frac{4\epsilon}{8\pi^2} \int_M Q u dV_g + \epsilon \log \int_M \tilde{Q} e^{4u} dV_g \\ &= II(u) - \frac{4\epsilon}{8\pi^2} \int_M Q u dV_g + \epsilon \log \int_M \tilde{Q} e^{4u} dV_g. \end{aligned}$$

Let u_k satisfy

$$\lim_{k \rightarrow +\infty} II(u_k) = \inf II.$$

Then for any $\epsilon > 0$ and fixed u_k , we have

$$\inf II_\epsilon \leq II_\epsilon(u_k) = II(u_k) - \frac{4\epsilon}{8\pi^2} \int_M Q_g u_k dV_g + \epsilon \log \int_M \tilde{Q} e^{4u_k}.$$

Letting $\epsilon \rightarrow 0$, we get

$$\overline{\lim}_{\epsilon \rightarrow 0} (\inf II_\epsilon) \leq II(u_k).$$

Then letting $k \rightarrow +\infty$, we get

$$\overline{\lim}_{\epsilon \rightarrow 0} (\inf II_\epsilon) \leq \inf II.$$

Next, we prove

$$\underline{\lim}_{\epsilon \rightarrow 0} (\inf II_\epsilon) \geq \inf II. \quad (2.3)$$

Let u_ϵ attain $\inf II_\epsilon$. Since $II_\epsilon(u+c) = II_\epsilon(u)$, we may assume $\int_M u_\epsilon dV_g = 0$. Obviously,

$$II_\epsilon(u_\epsilon) = \left(1 - \frac{\epsilon}{8\pi^2}\right) II(u_\epsilon) + \frac{\epsilon}{8\pi^2} \int_M u_\epsilon P_g u_\epsilon.$$

By (1.6), we have

$$\inf II_\epsilon = II_\epsilon(u_\epsilon) \geq \left(1 - \frac{\epsilon}{8\pi^2}\right) II(u_\epsilon) \geq \left(1 - \frac{\epsilon}{8\pi^2}\right) \inf II.$$

Letting $\epsilon \rightarrow 0$, we get (2.3). \square

Now let u_ϵ be the minimum point of II_ϵ . It is clear that u_ϵ satisfies the following equation:

$$\begin{cases} P_g u_\epsilon + 2 \left(1 - \frac{\epsilon}{8\pi^2}\right) Q_g = 2 \left(1 - \frac{\epsilon}{8\pi^2}\right) \tilde{Q} e^{4u_\epsilon} \\ \int_M \tilde{Q} e^{4u_\epsilon} dV_g = 8\pi^2. \end{cases}$$

The same proof of Lemma 2.3 in [14] yields the following.

Lemma 2.3. *There are constants $C_1(q)$, $C_2(q)$, $C_3(q)$ depending only on p and M such that, for r sufficiently small and for any $x \in M$ there holds*

$$\int_{B_r(x)} |\nabla^3 u_\epsilon|^q dV_g \leq C_1(q) r^{4-3q}, \quad \int_{B_r(x)} |\nabla^2 u_\epsilon|^q dV_g \leq C_2(q) r^{4-2q},$$

and

$$\int_{B_r(x)} |\nabla u_\epsilon|^q dV_g \leq C_3(q) r^{4-q}$$

where, respectively, $q < \frac{4}{3}$, $q < 2$, and $q < 4$.

3. The proof of Theorem 1.1

Let x_ϵ be the maximum point of u_ϵ . Assume $m_\epsilon = u_\epsilon(x_\epsilon)$, $r_\epsilon = e^{-m_\epsilon}$, and $x_\epsilon \rightarrow p$. Let $\{e_i(x)\}$ be an orthonormal basis of TM near p and $\exp_x : T_x M \rightarrow M$ be the exponential mapping. The smooth mapping $E : B_\delta(p) \times B_r \rightarrow M$ is defined as follows,

$$E(x, y) = \exp_x(y^i e_i(x)),$$

where B_r is a small ball in \mathbb{R}^n . Note that $E(x, \cdot) : T_x M \rightarrow M$ are all differential homeomorphism if r is sufficiently small.

We set

$$g_{ij}(x, y) = \left\langle (\exp_x)_* \frac{\partial}{\partial y^i}, (\exp_x)_* \frac{\partial}{\partial y^j} \right\rangle_{E(x, y)}.$$

It is well-known that $g = (g_{ij})$ is smooth, and $g(x, y) = I + O(|y|^2)$ for any fixed x . That is, we are able to find a constant K , s.t.

$$\|g(x, y) - I\|_{C^0(B_\delta(p) \times B_r)} \leq K|y|^2$$

when δ and r are sufficiently small. Moreover, for any $\varphi \in C^\infty(B_\rho(x_k))$ we have

$$\Delta_g u_\epsilon = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^k} \left(\sqrt{|g|} g^{km} \frac{\partial u_\epsilon(E(x_\epsilon, x))}{\partial x^m} \right),$$

$$|\nabla u_\epsilon|^2 = g^{pq} \frac{\partial u_\epsilon(E(x_\epsilon, x))}{\partial x^p} \frac{\partial u_\epsilon(E(x_\epsilon, x))}{\partial x^q},$$

and

$$\int_{B_\delta(x_k)} \varphi dV_g = \int_{E^{-1}(x_k, y) \cap B_\delta(x_k)} \varphi(E^{-1}(x_k, y)) \sqrt{|g|} dy.$$

We define

$$\tilde{u}_\epsilon(x) = u_\epsilon(E(x_\epsilon, x)),$$

and

$$v_\epsilon(x) = \tilde{u}_\epsilon(r_\epsilon x), \quad v'_\epsilon = v_\epsilon - m_\epsilon. \quad (3.1)$$

Now v_ϵ, v'_ϵ are functions defined on $B_{\frac{r}{2r_\epsilon}} \subset \mathbb{R}^n$.

We have

$$\Delta_{g_\epsilon}^2 v'_\epsilon = r_\epsilon^2 O(|\nabla^2 v'_\epsilon|) + r_\epsilon^3 O(|\nabla v'_\epsilon|) + \tilde{Q}_g(E(x_\epsilon, r_\epsilon x)) e^{4v'_\epsilon}. \quad (3.2)$$

It follows from Lemma 2.3 that,

$$\|\nabla^2 v'_\epsilon\|_{L^q(B_L)} \leq C(L, q) \quad \text{and} \quad \|\nabla v'_\epsilon\|_{L^q(B_L)} \leq C'(L, q) \quad \text{for any } q \in (1, 2).$$

Then (3.2) implies that

$$\|\Delta_{g_\epsilon}(\Delta_{g_\epsilon} v'_\epsilon)\|_{L^q(B_L)} \leq C'(L).$$

Using the standard elliptic estimates, we get

$$\|\Delta_{g_k} v'_\epsilon\|_{W^{2,q}(B_L)} \leq C_2(L).$$

The Sobolev inequality then yields,

$$\|\Delta_{g_\epsilon} v'_\epsilon\|_{L^q(B_L)} \leq C_3(q, L) \quad \text{for any } q \in (0, 4).$$

We therefore have

$$\|v'_\epsilon\|_{W^{2,q}(B_L)} \leq C_4(L).$$

Hence, by using the standard elliptic estimates, we see that v'_ϵ converge smoothly to w , which satisfies

$$\Delta_0^2 w = 2\tilde{Q}(p)e^{4w},$$

where Δ_0 is the Laplace operator in \mathbb{R}^4 . Moreover, it is easy to check that

$$\int_{B_L} \tilde{Q}(p)e^{4w} dx \leq 8\pi^2$$

for any $L > 0$. By the result of [13], we have

(a) $w = -\log(1 + \frac{\sqrt{3\tilde{Q}(p)}}{12}|x|^2)$, with

$$\tilde{Q}(p) \int_{\mathbb{R}^4} e^{4w} dV_g = 8\pi^2,$$

or

(b) w has the following asymptotic behavior:

$$-\Delta w \rightarrow a > 0 \quad \text{as } |x| \rightarrow +\infty.$$

We claim that (b) does not happen. If it does, then we have

$$\lim_{\epsilon \rightarrow +0} \int_{B_R} -\Delta_g v_\epsilon \sim \frac{\omega_3}{4} a R^4.$$

However, it follows from Lemma 2.3 that

$$\int_{B_R} |\Delta_{g_\epsilon} v'_\epsilon| dV_g \leq C R^2.$$

This shows that the case (b) does not happen.

For simplicity, let $\lambda = \frac{\sqrt{3\tilde{Q}(p)}}{12}$, so that we have

$$w = -\log(1 + \lambda|x|^2).$$

Now, we consider the convergence of u_ϵ outside the bubble. By Lemma 2.3, u_ϵ is bounded in $W^{3,q}$ for any $q < \frac{4}{3}$. Then, it is easy to check that $u_\epsilon - \bar{u}_\epsilon \rightarrow G_p$, where $\bar{u}_\epsilon = \frac{1}{|M|} \int_M u_\epsilon dV_g$ and

$$P_g G_p + 2Q_g = 16\pi^2 \delta_p, \quad \int_M G_p dV_g = 0.$$

To prove the strong convergence of $u_\epsilon - \bar{u}_\epsilon$, we first show the following lemma.

Lemma 3.1. *Given $\Omega \subset\subset M \setminus \{p\}$, there holds*

$$\int_\Omega e^{q(u_\epsilon - \bar{u}_\epsilon)} dV_g < C(\Omega, q)$$

for any $q > 0$.

Proof. Let $f_\epsilon = \tilde{Q}_g e^{4u_\epsilon}$. For any $x \in \Omega$, we have the following representation formula,

$$u_\epsilon(x) - \bar{u}_\epsilon = - \int_M G(x, y) Q_g dV_{g,y} + \int_M G(x, y) f_\epsilon.$$

Hence, if we let $\Omega_\epsilon = M \setminus B_{Lr_\epsilon}(x_\epsilon)$, and $\mu_\epsilon = 1 / \int_{\Omega_\epsilon} |f_\epsilon| dV_g$, we have, for any $q' > 0$,

$$e^{q' \mu_\epsilon (u_\epsilon - \bar{u}_\epsilon + \int_M G(x, y) Q_g dV_g)} = e^{\int_{\Omega_\epsilon} q' G(x, y) \mu_\epsilon f_\epsilon(y) dV_{g,y} + \int_{B_{Lr_\epsilon}} q' G(x, y) \mu_\epsilon f_\epsilon(y) dV_{g,y}}.$$

Notice that for any $x \in \Omega$ and $y \in B_{Lr_\epsilon}(x_\epsilon)$, $|G(x, y)| < C(\Omega, L)$. We have

$$\int_{B_{Lr_\epsilon}(x_\epsilon)} q' |G(x, y)| \mu_\epsilon f_\epsilon(y) dV_{g,y} \leq C_1(L) \int_{B_{Lr_\epsilon}(x_\epsilon)} f_\epsilon(y) dV_g \leq C_2(L),$$

and

$$e^{\int_{\Omega_\epsilon} q' G(x,y) \mu_\epsilon f_\epsilon(y) dV_{g,y}} \leq \int_{\Omega_\epsilon} \frac{f_\epsilon(y)}{\|f_\epsilon\|_{L^1(\Omega_\epsilon)}} e^{q' G(x,y)} dV_{g,y}.$$

Therefore, by using Jensen's inequality and Fubini's theorem, we obtain

$$\begin{aligned} \int_{\Omega} e^{\int_{\Omega_\epsilon} q' G(x,y) \mu_\epsilon f_\epsilon(y) dV_{g,y}} dV_g &\leq \int_{\Omega} \frac{f_\epsilon(y)}{\|f_\epsilon\|_{L^1(\Omega_\epsilon)}} \left(\int_{\Omega_\epsilon} e^{q' G(x,y)} dV_{g,x} \right) dV_{g,y} \\ &\leq C \int_{\Omega} \frac{f_\epsilon(y)}{\|f_\epsilon\|_{L^1(\Omega_\epsilon)}} \left(\int_{\Omega_\epsilon} \frac{1}{|x-y|^{\frac{q'}{8\pi^2}}} dV_{g,x} \right) dV_{g,y}. \end{aligned}$$

The last integral is finite provided $q' < 32\pi^2$. Hence, for any $q > 0$, if ϵ is sufficiently small so that $q \leq q' \mu_\epsilon$ we have

$$\begin{aligned} \int_{\Omega} e^{q(u_\epsilon(x) - \bar{u}_\epsilon)} dx &\leq \int_{\Omega} e^{q' \mu_\epsilon(u_\epsilon(x) - \bar{u}_\epsilon)} dx \\ &\leq C \int_{\Omega} e^{\int_{\Omega_\epsilon} q' G(x,y) \mu_\epsilon f_\epsilon(y) dV_{g,y}} dV_g \leq C. \quad \square \end{aligned}$$

As a consequence of the above lemma, we have the following lemma.

Lemma 3.2. *Let $\Omega \subset \subset M \setminus \{x_0\}$. Then $u_\epsilon - \bar{u}_\epsilon$ converges to G_{x_0} in $C^k(\Omega)$ as $\epsilon \rightarrow 0$.*

Proof. It is easy to see that $\bar{u}_\epsilon < C$. Then the lemma follows. \square

Remark 3.3. In B_{δ_0} , using the above coordinates, we set $p = y_\epsilon$ for any ϵ . Clearly, $y_\epsilon \rightarrow 0$. Then we also have $u_\epsilon(E(p, x)) - \bar{u}_\epsilon \rightarrow G_p(E(p, x))$. Moreover, we may write

$$G(E(p, x)) = -2 \log |x| + S_0(p) + S_1(x),$$

where $S_0(p)$ is a constant and $S_1 = O(r)$. It is easy to check $\tilde{u}_\epsilon - \bar{u}_\epsilon \rightarrow G(E(p, x))$ smoothly in $B_{\delta_0} \setminus B_\delta$ for any fixed δ .

Now, we give a lower bound of $\lim_{\epsilon \rightarrow 0} \int_M \langle u_\epsilon, u_\epsilon \rangle dV_g$. We write

$$\int_M \langle u_\epsilon, u_\epsilon \rangle dV_g = I_1 + I_2 + I_3,$$

where I_1, I_2, I_3 denote the integrals on $M \setminus B_\delta(x_\epsilon)$, $B_{Lr_\epsilon}(x_\epsilon)$ and $B_\delta \setminus B_{Lr_\epsilon}(x_\epsilon)$ (any fixed L and δ) respectively. We remark that the integral I_1, I_2 can be easily treated due to the above lemmas. On the other hand, by Lemma 2.3, we have

$$\int_{B_\delta \setminus B_{Lr_\epsilon}(x_\epsilon)} |\nabla_g u_\epsilon|^2 dV_g \rightarrow \int_{B_\delta(p)} |\nabla_g G|^2 = O(\delta^2).$$

So, the key point is to calculate

$$\int_{B_\delta(x_\epsilon) \setminus B_{Lr_\epsilon}(x_\epsilon)} |\Delta_g u_\epsilon|^2 dV_g.$$

We are going to prove the following lemma.

Lemma 3.4. *We have*

$$\int_{B_\delta(x_\epsilon) \setminus B_{Lr_\epsilon}(x_\epsilon)} |\Delta_g u_\epsilon|^2 dV_g \geq \int_{B_\delta \setminus B_{Lr_\epsilon}} |(1 - b|x|^2) \Delta_0 \tilde{u}_\epsilon|^2 dx + J(L, \epsilon, \delta),$$

for some $b > 0$, where

$$\lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} J(L, \epsilon, \delta) = 0.$$

Proof. Since we have

$$\begin{aligned} |\Delta_g u_\epsilon|^2 &= \left| g^{km} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^k \partial x^m} + O(|\nabla \tilde{u}_\epsilon|) \right|^2 \\ &= \left| g^{km} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^k \partial x^m} \right|^2 + O(|\nabla^2 \tilde{u}_\epsilon| (|\nabla \tilde{u}_\epsilon|)) + O(|\nabla \tilde{u}_\epsilon|^2), \end{aligned}$$

and since $\tilde{u}_\epsilon - \bar{u}_\epsilon$ converges to $G_p(E(p, x))$ in $W^{3,q}$ for any $q < \frac{4}{3}$, we get

$$\begin{aligned} &\int_{B_\delta \setminus B_{Lr_\epsilon}} O(|\nabla^2 \tilde{u}_\epsilon| (|\nabla \tilde{u}_\epsilon|) + O(|\nabla \tilde{u}_\epsilon|^2)) \\ &\leq C(\|\nabla^2 G_p\|_{L^q(B_\delta \setminus B_{Lr_\epsilon})} \|\nabla G_p\|_{L^{q'}(B_\delta \setminus B_{Lr_\epsilon})} + \|G_p\|_{W^{1,2}(B_\delta \setminus B_{Lr_\epsilon})}) \\ &= J(L, \epsilon, \delta), \end{aligned}$$

where $\frac{3}{2} < q < 2$, and $\frac{1}{q'} + \frac{1}{q} = 1$.

Let $g^{km} = \delta^{km} + A^{km}$, with $|A^{km}| \leq K|x|^2$ for any ϵ, k, m . Consequently, we have

$$\left| g^{km} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^k \partial x^m} \right|^2 = |\Delta_0 \tilde{u}_\epsilon|^2 + 2 \sum_{s,t} A^{st} \Delta_0 \tilde{u}_\epsilon \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t} + \sum_{k,m,s,t} A^{km} A^{st} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^k \partial x^m} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t}.$$

It is clear that

$$2 \int_{B_\delta \setminus B_{Lr_\epsilon}} \left| A^{st} \Delta_0 \tilde{u}_\epsilon \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t} \right| \leq K \int_{B_\delta \setminus B_{Lr_\epsilon}} \left(|x|^2 |\Delta_0 \tilde{u}_\epsilon|^2 + |x|^2 \left| \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t} \right|^2 \right) dx,$$

and

$$\begin{aligned} \int_{B_\delta \setminus B_{Lr_\epsilon}} |x|^2 \left| \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t} \right|^2 dx &= \int_{B_\delta \setminus B_{Lr_\epsilon}} |x|^2 \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^t \partial x^t} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^s} dx \\ &\quad + \int_{B_\delta \setminus B_{Lr_\epsilon}} O(|x| |\nabla \tilde{u}_\epsilon| |\nabla^2 \tilde{u}_\epsilon|) dx \\ &\quad + \int_{\partial(B_\delta \setminus B_{Lr_\epsilon})} |x|^2 \frac{\partial \tilde{u}_\epsilon}{\partial x^t} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t} \left\langle \frac{\partial}{\partial x^t}, \frac{\partial}{\partial r} \right\rangle ds \\ &\quad + \int_{\partial(B_\delta \setminus B_{Lr_\epsilon})} |x|^2 \left(\frac{\partial \tilde{u}_\epsilon}{\partial x^t} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^s} \left\langle \frac{\partial}{\partial x^s}, \frac{\partial}{\partial r} \right\rangle \right) ds \\ &= \int_{B_\delta \setminus B_{Lr_\epsilon}} |x|^2 \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^t \partial x^t} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^s} dx + J(L, \epsilon, \delta). \end{aligned}$$

On $\partial B_\delta(x_\epsilon)$, since $\tilde{u}_\epsilon - \bar{u}_\epsilon \rightarrow G_p(E(p, x))$, as $\epsilon \rightarrow 0$, we have

$$\begin{aligned} & \int_{\partial B_\delta} |x|^2 \frac{\partial \tilde{u}_\epsilon}{\partial x^i} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^j \partial x^k} \left\langle \frac{\partial}{\partial x^s}, \frac{\partial}{\partial r} \right\rangle ds \\ & \rightarrow \int_{\partial B_\delta} |x|^2 \left(\frac{\partial G_p(E(p, x))}{\partial x^i} \frac{\partial^2 G_p(E(p, x))}{\partial x^j \partial x^k} \left\langle \frac{\partial}{\partial x^s}, \frac{\partial}{\partial r} \right\rangle \right) ds \\ & = \int_{\partial B_\delta} O\left(\frac{1}{\delta}\right) ds \\ & = O(\delta^2). \end{aligned}$$

On ∂B_{Lr_ϵ} , since $\tilde{u}_\epsilon(r_\epsilon x) - m_\epsilon \rightarrow \omega$ as $\epsilon \rightarrow 0$, we have

$$\frac{1}{r_\epsilon^2} \int_{\partial B_{Lr_\epsilon}} |x|^2 \frac{\partial \tilde{u}_\epsilon}{\partial x^i} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^j \partial x^k} \left\langle \frac{\partial}{\partial x^s}, \frac{\partial}{\partial r} \right\rangle ds \rightarrow \int_{\partial B_L} |x|^2 \frac{\partial \omega}{\partial x^i} \frac{\partial^2 \omega}{\partial x^j \partial x^k} \left\langle \frac{\partial}{\partial x^s}, \frac{\partial}{\partial r} \right\rangle ds.$$

Then we get

$$\lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{\partial(B_\delta \setminus B_{Lr_\epsilon})} |x|^2 \frac{\partial \tilde{u}_\epsilon}{\partial x^i} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^j \partial x^k} \left\langle \frac{\partial}{\partial x^s}, \frac{\partial}{\partial r} \right\rangle ds = 0.$$

Moreover,

$$2 \sum_{k,s,t} \int_{B_\delta \setminus B_{Lr_\epsilon}} \left| A^{st} \Delta_0 \tilde{u}_\epsilon \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t} \right| \leq 4K \int_{B_\delta \setminus B_{Lr_\epsilon}} |x|^2 |\Delta_0 \tilde{u}_\epsilon|^2 dx + J(L, \epsilon, \delta).$$

A similar argument as above then gives

$$\int_{B_\delta \setminus B_{Lr_\epsilon}} \sum_{k,m,s,t} A^{km} A^{st} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^k \partial x^m} \frac{\partial^2 \tilde{u}_\epsilon}{\partial x^s \partial x^t} \leq K^2 \int_{B_\delta \setminus B_{Lr_\epsilon}} |x|^4 |\Delta_0 \tilde{u}_\epsilon|^2 dx + J(L, \epsilon, \delta).$$

This proves the lemma. \square

Lemma 3.5. *There is a function sequence $U_\epsilon \in W^{2,2}(B_\delta \setminus B_{Lr_\epsilon})$ s.t.*

$$\begin{aligned} U_\epsilon|_{\partial B_\delta} &= -2 \log \delta + S_0(p) + \bar{u}_\epsilon, & U_\epsilon|_{\partial B_{Lr_\epsilon}} &= w(L) + m_\epsilon \\ \frac{\partial U_\epsilon}{\partial r} \Big|_{\partial B_\delta} &= -\frac{2}{\delta}, & \frac{\partial U_\epsilon}{\partial r} \Big|_{\partial B_{Lr_\epsilon}} &= w'(L) \end{aligned}$$

and

$$\begin{aligned} & \int_{B_\delta \setminus B_{Lr_\epsilon}} \left| \Delta_0 \left((1 - b|x|^2)(U_\epsilon - \bar{u}_\epsilon) \right) \right|^2 dx \\ &= \int_{B_\delta \setminus B_{Lr_\epsilon}} |(1 - b|x|^2) \Delta_0 \tilde{u}_\epsilon|^2 dx + J(L, \epsilon, \delta), \end{aligned}$$

where

$$\lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} J(L, \epsilon, \delta) = 0.$$

Proof. Let u'_ϵ be the solution of

$$\begin{cases} \Delta_0^2 u'_\epsilon = \Delta_0^2 v_\epsilon \\ \frac{\partial u'_\epsilon}{\partial n} \Big|_{\partial B_{2L}} = \frac{\partial v_\epsilon}{\partial n} \Big|_{\partial B_{2L}}, & u'_\epsilon|_{\partial B_{2L}} = v_\epsilon|_{\partial B_{2L}} \\ \frac{\partial u'_\epsilon}{\partial n} \Big|_{\partial B_L} = \frac{\partial w}{\partial n} \Big|_{\partial B_L}, & u'_\epsilon|_{\partial B_L} = m_\epsilon + w|_{\partial B_L}, \end{cases}$$

where v_ϵ is defined by (3.1). We set

$$U'_\epsilon = \begin{cases} u'_\epsilon \left(\frac{x}{r_\epsilon} \right) & Lr_\epsilon \leq |x| \leq 2Lr_\epsilon \\ \tilde{u}_\epsilon(x) & 2Lr_\epsilon \leq |x|. \end{cases}$$

It is easy to see that $u'_\epsilon - m_\epsilon$ converges to w smoothly on $B_{2L} \setminus B_L$; then we have

$$\lim_{\epsilon \rightarrow 0} \int_{B_{2Lr_\epsilon} \setminus B_{Lr_\epsilon}} (1 - b|x|^2)^2 (|\Delta_0 U'_\epsilon|^2 - |\Delta_0 \tilde{u}_\epsilon|^2) dx = 0.$$

Let η be a smooth function which satisfies:

$$\eta(t) = \begin{cases} 1 & t \leq 1/2 \\ 0 & t > 2/3. \end{cases}$$

Set $G_\epsilon = \eta\left(\frac{|x|}{\delta}\right)(\tilde{u}_\epsilon - S_0(p) + 2 \log |x|^2 - \bar{u}_\epsilon) - 2 \log |x|^2 + S_0(p)$. Recall that $u_\epsilon - \bar{u}_\epsilon$ converges to G_p smoothly on $M \setminus B_{\frac{\delta}{2}}(p)$; then we have

$$\begin{aligned} G_\epsilon &\rightarrow -2 \log |x|^2 + S_0(p) + \eta\left(\frac{|x|}{\delta}\right) S_1(x), \\ \tilde{u}_\epsilon - G_\epsilon - \bar{u}_\epsilon &\rightarrow \left(\eta\left(\frac{|x|}{\delta}\right) - 1 \right) S_1(x). \end{aligned}$$

Therefore

$$\begin{aligned} &\lim_{\epsilon \rightarrow 0} \left| \int_{B_\delta \setminus B_{\delta/2}} |\Delta_0 \tilde{u}_\epsilon|^2 dx - \int_{B_\delta \setminus B_{\delta/2}} |\Delta_0 G_\epsilon|^2 dx \right| \\ &= \left| \int_{B_\delta \setminus B_{\delta/2}} |\Delta_0 G_p(E(p, x))|^2 dx - \int_{B_\delta \setminus B_{\delta/2}} |\Delta_0 G_\epsilon|^2 dx \right| \\ &= \left| \int_{B_\delta \setminus B_{\delta/2}} \Delta_0 G_p(E(p, x) + G_\epsilon) dx \int_{B_\delta \setminus B_{\delta/2}} \Delta_0 (G_0(E(p, x)) - G_\epsilon) dx \right| \\ &\leq \sqrt{\int_{B_\delta \setminus B_{\delta/2}} \left| \Delta_0 \left(\eta\left(\frac{|x|}{\delta}\right) - 1 \right) S_1(x) \right|^2 dx \int_{B_\delta \setminus B_{\delta/2}} \left| \Delta_0 \left(G_p - 2 \log |x|^2 + \eta\left(\frac{|x|}{\delta}\right) S_1(x) \right) \right|^2 dx} \\ &\leq C\sqrt{\delta |\log \delta|}. \end{aligned}$$

Now set

$$U_\epsilon = \begin{cases} U'_\epsilon(x) & |x| \leq \frac{\delta}{2} \\ G_\epsilon(x) + \bar{u}_\epsilon & \delta/2 \leq |x| \leq \delta. \end{cases}$$

We then have,

$$\begin{aligned} \int_{B_\delta \setminus B_{L\epsilon}} |(1 - B|x|^2) \Delta_0(U_\epsilon - \bar{u}_\epsilon)|^2 dx &= \int_{B_\delta \setminus B_{L\epsilon}} |\Delta_0(1 - B|x|^2)(U_\epsilon - \bar{u}_\epsilon)|^2 dx \\ &+ \int_{B_\delta \setminus B_{L\epsilon}} O(|\nabla U_\epsilon|^2 + |U_\epsilon - \bar{u}_\epsilon|^2) dV_g. \end{aligned}$$

To complete the proof, we only need to prove

$$\lim_{L \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \|U_\epsilon - \bar{u}_\epsilon\|_{W^{1,2}(B_\delta \setminus B_{L\epsilon})} = 0. \quad (3.3)$$

We have

$$\begin{aligned} \|U_\epsilon - \bar{u}_\epsilon\|_{W^{1,2}(B_\delta \setminus B_{L\epsilon})}^2 &= \|U_\epsilon - \bar{u}_\epsilon\|_{W^{1,2}(B_\delta \setminus B_{\delta/2})}^2 + \|U_\epsilon - \bar{u}_\epsilon\|_{W^{1,2}(B_{\delta/2} \setminus B_{2L\epsilon})}^2 \\ &+ \|U_\epsilon - \bar{u}_\epsilon\|_{W^{1,2}(B_{\delta/2} \setminus B_{2L\epsilon})}^2 + \|U_\epsilon - \bar{u}_\epsilon\|_{W^{1,2}(B_{2L\epsilon} \setminus B_{L\epsilon})}^2 \\ &= \|G_\epsilon\|_{W^{1,2}(B_\delta \setminus B_{\delta/2})}^2 + \|\tilde{u}_\epsilon - \bar{u}_\epsilon\|_{W^{1,2}(B_{\delta/2} \setminus B_{2L\epsilon})}^2 \\ &+ \|\tilde{u}_\epsilon - \bar{u}_\epsilon\|_{W^{1,2}(B_{2L\epsilon} \setminus B_{L\epsilon})}^2 + \|U'_\epsilon - \tilde{u}_\epsilon\|_{W^{1,2}(B_{2L\epsilon} \setminus B_{L\epsilon})}^2 \\ &\leq \|G_\epsilon\|_{W^{1,2}(B_\delta \setminus B_{\delta/2})}^2 + \|\tilde{u}_\epsilon - \bar{u}_\epsilon\|_{W^{1,2}(B_{\delta/2})}^2 \\ &+ \|U'_\epsilon - \tilde{u}_\epsilon\|_{W^{1,2}(B_{2L\epsilon} \setminus B_{L\epsilon})}^2. \end{aligned}$$

It is easy to check that

$$\lim_{\epsilon \rightarrow 0} \|U'_\epsilon - \tilde{u}_\epsilon\|_{W^{1,2}(B_{2L\epsilon} \setminus B_{L\epsilon}(x_\epsilon))}^2 = 0.$$

Recall $\tilde{u}_\epsilon - \bar{u}_\epsilon \rightarrow G_p(E(p, x))$. We get (3.3). \square

Now, we are going to apply capacity estimate to derive the lower bound for

$$\int_{B_\delta \setminus B_{L\epsilon}} \left| \Delta_0 \left((1 - b|x|^2)(U_\epsilon - \bar{u}_\epsilon) \right) \right|^2 dx.$$

First we need to calculate

$$\inf_{\substack{\Phi|_{\partial B_r} = P_1, \Phi|_{\partial B_R} = P_2, \frac{\partial \Phi}{\partial r} \Big|_{\partial B_r} = Q_1, \frac{\partial \Phi}{\partial r} \Big|_{\partial B_R} = Q_2}} \int_{B_R \setminus B_r} |\Delta_0 \Phi|^2 dx,$$

where P_1, P_2, Q_1, Q_2 are constants. Obviously, the minimum can be attained by the function Φ which satisfies

$$\begin{cases} \Delta_0^2 \Phi = 0 \\ \Phi|_{\partial B_r} = P_1, & \Phi|_{\partial B_R} = P_2, & \frac{\partial \Phi}{\partial r} \Big|_{\partial B_r} = Q_1, & \frac{\partial \Phi}{\partial r} \Big|_{\partial B_R} = Q_2. \end{cases}$$

Clearly, we can set

$$\Phi = A \log r + Br^2 + \frac{C}{r^2} + D,$$

where A, B, C, D are all constants. Then we have

$$\begin{cases} A \log r + Br^2 + \frac{C}{r^2} + D = P_1 \\ A \log R + BR^2 + \frac{C}{R^2} + D = P_2 \\ \frac{A}{r} + 2Br - 2\frac{C}{r^3} = Q_1 \\ \frac{A}{R} + 2BR - 2\frac{C}{R^3} = Q_2. \end{cases}$$

We have

$$\begin{cases} A = \frac{P_1 - P_2 + \frac{\varrho}{2}rQ_1 + \frac{\varrho}{2}RQ_2}{\log r/R + \varrho} \\ B = \frac{-2P_1 + 2P_2 - rQ_1 \left(1 + \frac{2r^2}{R^2 - r^2} \log r/R\right) + RQ_2 \left(1 + \frac{2R^2}{R^2 - r^2} \log r/R\right)}{4(R^2 + r^2)(\log r/R + \varrho)}, \end{cases}$$

where $\varrho = \frac{R^2 - r^2}{R^2 + r^2}$. Furthermore,

$$\int_{B_R \setminus B_r} |\Delta_0 \Phi|^2 dx = -8\pi^2 A^2 \log r/R + 32\pi^2 AB(R^2 - r^2) + 32\pi^2 B^2(R^4 - r^4).$$

In our case, $R = \delta, r = Lr_\epsilon$,

$$P_1 = (1 - B|x|^2)U_\epsilon|_{\partial B_{Lr_\epsilon}} = m_\epsilon - \bar{u}_\epsilon + w(L) + O(r_\epsilon \bar{u}_\epsilon),$$

$$P_2 = (1 - B|x|^2)U_\epsilon|_{\partial B_\delta} = -2 \log \delta + S_0(p) + O(\delta \log \delta),$$

$$Q_1 = \frac{\partial(1 - B|x|^2)U_\epsilon}{\partial r} \Big|_{\partial B_{Lr_\epsilon}} = \frac{2\lambda L}{r_\epsilon(1 + \lambda L^2)},$$

$$Q_2 = \frac{\partial(1 - B|x|^2)U_\epsilon}{\partial r} \Big|_{\partial B_\delta} = -\frac{2}{\delta} + O(\delta \log \delta).$$

If we define

$$\begin{aligned} N(L, \epsilon, \delta) &= w(L) + 2 \log \delta - S_0 - \frac{\varrho}{2} \frac{2\lambda L^2}{1 + \lambda L^2} \\ &= w(L) + 2 \log \delta - S_0 - 2 + O(\delta \log \delta) + O\left(\frac{1}{L^2}\right) + O(Lr_\epsilon), \end{aligned}$$

and

$$P = \log \delta - \log L,$$

then we have

$$\begin{aligned} A^2(-\log Lr_\epsilon/\delta) &= \left(\frac{m_\epsilon - \bar{u}_\epsilon + N(L, \epsilon, \delta)}{m_\epsilon + P - \varrho}\right)^2 (m_\epsilon + P) \\ &= \left(1 + \frac{P - \varrho}{m_\epsilon}\right)^{-2} \left(1 + \frac{P}{m_\epsilon}\right) m_\epsilon \left(1 - \frac{\bar{u}_\epsilon}{m_\epsilon} + \frac{N(L, \epsilon, \delta)}{m_\epsilon}\right)^2 \end{aligned}$$

$$\begin{aligned}
&= \left(1 - 2\frac{P - \varrho}{m_\epsilon} + O\left(\frac{1}{m_\epsilon^2}\right)\right) \left(1 + \frac{P}{m_\epsilon}\right) m_\epsilon \\
&\quad \times \left[\left(1 - \frac{\bar{u}_\epsilon}{m_\epsilon}\right)^2 + 2\left(1 - \frac{\bar{u}_\epsilon}{m_\epsilon}\right) \frac{N(L, \epsilon, \delta)}{m_\epsilon} \right. \\
&\quad \left. + O\left(\frac{1}{m_\epsilon^2}\right) + O(e^{-m_\epsilon} m_\epsilon) \frac{\bar{u}_\epsilon}{m_\epsilon} \right] \\
&= m_\epsilon \left(1 - \frac{\bar{u}_\epsilon}{m_\epsilon}\right)^2 + 2\left(1 - \frac{\bar{u}_\epsilon}{m_\epsilon}\right) N(L, \epsilon, \delta) \\
&\quad - (P - 2\varrho) \left(1 - \frac{\bar{u}_\epsilon}{m_\epsilon}\right)^2 + O\left(\frac{1}{m_\epsilon}\right) \left(1 - \frac{\bar{u}_\epsilon}{m_\epsilon}\right)^2 + O\left(\frac{1}{m_\epsilon}\right),
\end{aligned}$$

and

$$\begin{aligned}
A &= -\frac{m_\epsilon - \bar{u}_\epsilon + N(L, \epsilon, \delta)}{m_\epsilon - \log L + \log \delta + \varrho} = -\left(1 - O\left(\frac{1}{m_\epsilon}\right)\right)^{-1} \left(1 - \frac{\bar{u}_\epsilon}{m_\epsilon} + O\left(\frac{1}{m_\epsilon}\right)\right) \\
&= -1 + \frac{\bar{u}_\epsilon}{m_\epsilon} + O\left(\frac{1}{m_\epsilon}\right).
\end{aligned}$$

Notice that $r_\epsilon m_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$, we have

$$\begin{aligned}
B &= \frac{-2m_\epsilon + 2\bar{u}_\epsilon + O(1) + \left(2\frac{\delta^2}{\delta^2 - (Lr_\epsilon)^2} + O(\delta \log \delta)\right) m_\epsilon}{4(\delta^2 + (Lr_\epsilon)^2)(\log L - m_\epsilon - \log \delta + \varrho)} \\
&= -\frac{1}{2\delta^2} \left(1 + \frac{\bar{u}_\epsilon}{m_\epsilon} + O\left(\frac{1}{m_\epsilon}\right)\right) \left(1 - O\left(\frac{1}{m_\epsilon}\right)\right)^{-1} \\
&= -\frac{1}{2\delta^2} \left(1 + \frac{\bar{u}_\epsilon}{m_\epsilon} + O\left(\frac{1}{m_\epsilon}\right)\right).
\end{aligned}$$

It concludes that

$$\begin{aligned}
&\int_{B_\delta \setminus B_{Lr_\epsilon}} |\Delta_0(1 - b|x|^2)(U_\epsilon - \bar{u}_\epsilon)|^2 dx \\
&\geq 8\pi^2 m_\epsilon \left(1 - \frac{\bar{u}_\epsilon}{m_\epsilon}\right)^2 + 16\pi^2 \left(1 - \frac{\bar{u}_\epsilon}{m_\epsilon}\right) N(L, \epsilon, \delta) - 8\pi^2 (P - 2\varrho) \left(1 - \frac{\bar{u}_\epsilon}{m_\epsilon}\right)^2 \\
&\quad + 16\pi^2 \left(1 - \frac{\bar{u}_\epsilon}{m_\epsilon}\right) \left(1 + \frac{\bar{u}_\epsilon}{m_\epsilon}\right) + 8\pi^2 \left(1 + \frac{\bar{u}_\epsilon}{m_\epsilon}\right)^2 \\
&\quad + O\left(\frac{1}{m_\epsilon}\right) \left(1 - \frac{\bar{u}_\epsilon}{m_\epsilon}\right)^2 + O\left(\frac{1}{m_\epsilon}\right) + J_6(L, \epsilon, \delta).
\end{aligned}$$

Using the fact that $\bar{u}_\epsilon \leq C$, we have

$$(8\pi^2 - \epsilon)\bar{u}_\epsilon > 8\pi^2 \bar{u}_\epsilon + \epsilon C.$$

Therefore

$$\begin{aligned} II_\epsilon(u_\epsilon) &\geq \int_{B_{Lr_\epsilon}(x_\epsilon)} |\Delta_g u_\epsilon|^2 dV_g + \int_{B_\delta \setminus B_{Lr_\epsilon}} |\Delta_0(1 - |B|^2)(U_\epsilon - \bar{u}_\epsilon)|^2 dx + 8\pi^2 \bar{u}_\epsilon \\ &\quad + \int_{M \setminus B_\delta(x_0)} \langle G_p, G_p \rangle + 4 \int_M \tilde{Q} G_p dV_g + J(L, \epsilon, \delta) \\ &\geq 8\pi^2(m_\epsilon + C_1) \left(1 + \frac{\bar{u}_\epsilon}{m_\epsilon}\right)^2 + C_2 \left(1 + \frac{\bar{u}_\epsilon}{m_\epsilon}\right) + C_3 \end{aligned}$$

where C_1, C_2, C_3 are some constants. Since $II_\epsilon(u_\epsilon) = \inf II_\epsilon < C' < \infty$, we must have $(1 + \frac{\bar{u}_\epsilon}{m_\epsilon}) \rightarrow 0$ as $\epsilon \rightarrow 0$, i.e. $\frac{\bar{u}_\epsilon}{m_\epsilon} \rightarrow -1$.

Consequently, we have

$$\begin{aligned} &\int_{B_\delta \setminus B_{Lr_\epsilon}} |\Delta_0(1 - |b|x|^2)(U_\epsilon - \bar{u}_\epsilon)|^2 dx + 8\pi^2 \bar{u}_\epsilon \\ &\geq 8\pi^2 m_\epsilon \left(1 + \frac{\bar{u}_\epsilon}{m_\epsilon}\right)^2 + 16\pi^2 N(L, \epsilon, \delta) \left(1 - \frac{\bar{u}_\epsilon}{m_\epsilon}\right) \\ &\quad - 8\pi^2(\log \delta - \log L - 2\rho) \left(1 - \frac{\bar{u}_\epsilon}{m_\epsilon}\right)^2 + J(L, \epsilon, \delta) \\ &\geq 16\pi^2 \left(1 - \frac{\bar{u}_\epsilon}{m_\epsilon}\right) N(L, \epsilon, \delta) - 8\pi^2(\log \delta - \log L - 2\rho) \left(1 - \frac{\bar{u}_\epsilon}{m_\epsilon}\right)^2 + J(L, \epsilon, \delta). \end{aligned} \tag{3.4}$$

Since we have

$$\Delta_0 w = \frac{4\lambda^2 |x|^2}{(1 + \lambda|x|^2)^2} - \frac{8\lambda}{1 + \lambda|x|^2},$$

a direct calculation yields that

$$\int_{B_L} |\Delta_0 w|^2 dx = 16\pi^2 \log(1 + \lambda L^2) + \frac{8\pi^2}{3} + O\left(\frac{\log L}{L^2}\right).$$

On the other hand, it is obvious to see that,

$$\int_{B_\delta(x_\epsilon)} |\nabla u_\epsilon|^2 \rightarrow \int_{B_\delta(x_\epsilon)} |\nabla G_p|^2 = O(\delta \log \delta), \tag{3.5}$$

and

$$\begin{aligned} &\int_{M \setminus B_\delta(x_0)} \langle G_p, G_p \rangle dV_g \\ &= \int_{M \setminus B_\delta(x_0)} G_p P_g G_p dV_g - \int_{\partial B_\delta} \frac{\partial G_p}{\partial r} \Delta_g G_p dV_g + \int_{\partial B_\delta} G_p \frac{\partial \Delta G_p}{\partial r} dV_g \\ &\quad + \int_{\partial B_\delta} \left(\frac{2}{3} R G \frac{\partial G}{\partial r} - 2G \text{Ric}(dG, dr) \right) dS_g \\ &= -2 \int_M Q_g G_p dV_g - 16\pi^2 + 16\pi^2(-2 \log \delta + S_0(p)) + O(\delta \log \delta). \end{aligned} \tag{3.6}$$

Together with [Lemmas 3.4 and 3.5, \(3.4\)–\(3.6\)](#), we have

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} II_{\epsilon} &\geq 32\pi^2 \lim_{\epsilon \rightarrow 0} N(L, \epsilon, \delta) - 32\pi^2(\log \delta - \log L - 2) + 16\pi^2 \log(1 + \lambda L^2) \\ &\quad + \frac{8\pi^2}{3} + (-2 \log \delta + S_0(p))16\pi^2 + 2 \int_M Q_g G_p dV_g - 8\pi^2 \log 8\pi^2 \\ &\quad + O(\delta \log \delta) + O\left(\frac{\log L}{L^2}\right) \\ &= -16\pi^2 \log \frac{1 + \lambda L^2}{L^2} + \frac{8\pi^2}{3} - 16\pi^2 S_0(p) - 16\pi^2 \\ &\quad + 2 \int_M Q_g G_p dV_g - 8\pi^2 \log 8\pi^2 + O(\delta \log \delta) + O\left(\frac{\log L}{L^2}\right). \end{aligned}$$

Letting first $\delta \rightarrow 0$, then $L \rightarrow +\infty$, we get

$$\lim_{\epsilon \rightarrow 0} II_{\epsilon} \geq -16\pi^2 \log \lambda - 8\pi^2 \log 8\pi^2 - 16\pi^2 S_0 + (8/3 - 16)\pi^2 + 2 \int_M Q_g G_p dV_g.$$

This shows the first part of [Theorem 1.1](#), that is

$$\inf_{u \in W^{2,2}(M)} II(u) \geq \inf_{p \in M} \Lambda_g(\tilde{Q}, p).$$

The second part

$$\inf_{u \in W^{2,2}(M)} II(u) \leq \inf_{p \in M} \Lambda_g(\tilde{Q}, p)$$

follows from the proof of [Theorem 1.2](#) in the next section.

To end this section, we will prove a conformal property of $\Lambda_g(\tilde{Q}, p)$.

Lemma 3.6. *Letting $\tilde{g} \in [g] : \tilde{g} = e^{2v}g$ for some $v \in C^\infty(M)$, we have*

$$II_{\tilde{g}}(u) = II_g(u + v) - \int_M \langle v, v \rangle dV_g - 4 \int_M Qv dV_g.$$

If we set

$$P_{\tilde{g}} \tilde{G}_y + 2Q_{\tilde{g}} = 16\pi^2 \delta_y,$$

then

$$\tilde{G}_y = G_y - v, \quad \text{and} \quad \tilde{S}_0(y) = S_0(y) + v(y).$$

Proof. Since $P_{\tilde{g}} = e^{-4v} P_g$, $2Q_{\tilde{g}} = e^{-4v} (P_g v + 2Q_g)$, we get

$$\begin{aligned} II_{\tilde{g}}(u) &= \int_M \langle u, u \rangle dV_g + 2 \int_M (P_g v + 2Q_g) u dV_g - 8\pi^2 \log \int_M \tilde{Q} e^{4(u+v)} dV_g \\ &= \int_M \langle u + v, u + v \rangle dV_g + 4 \int_M Q_g u dV_g \\ &\quad - 8\pi^2 \log \int_M \tilde{Q} e^{4(u+v)} dV_g - \int_M \langle v, v \rangle dV_g \\ &= II_g(u + v) - \int_M \langle v, v \rangle dV_g - 4 \int_M Qv dV_g. \end{aligned}$$

On the other hand, we have

$$P_{\tilde{g}}(G - v) + 2Q_{\tilde{g}} = e^{-4v}(P_g G + 2Q_g) = 16\pi^2 e^{-4v} \delta_{y,g} = 16\pi^2 \delta_{y,\tilde{g}}.$$

Since $\text{dist}_{\tilde{g}}(y, x) = e^{v(y)} \text{dist}_g(y, x) + O(\text{dist}_g(y, x))^2$, we have

$$\begin{aligned} \tilde{G}_y &= G_y - v \\ &= -2 \log \text{dist}_g(y, x) + S_0(y) - v(y) + O(\text{dist}(y, x)) \\ &= -2 \log \text{dist}_{\tilde{g}}(y, x) + v(y) + S_0(y) + O(\text{dist}(y, x)). \end{aligned}$$

Thus $\tilde{S}_0(y) = S_0(y) + v(y)$. \square

4. Testing function

In this section, we will construct a blow up sequence ϕ_ϵ s.t.

$$II(\phi_\epsilon) < \inf_{x \in M} \Lambda(x).$$

We use standard notation from [10]. In a normal geodesic coordinate system $\{x^i\}$, we denote

$$R_{ijkl} = \langle R(\partial_k, \partial_l) \partial_j, \partial_i \rangle, \quad R_{ij} = -g^{jk} R_{ijkl},$$

where R is the curvature operator, defined as follows,

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}.$$

Suppose that p' is a point such that $\Lambda(p') = \inf_{x \in M} \Lambda(x)$.

We know that, locally we have

$$\begin{aligned} g_{pq} &= \delta_{pq} + \frac{1}{3} R_{pijq}(p') x^i x^j + \frac{1}{6} R_{pijq,k}(p') x^i x^j x^k \\ &\quad + \left(\frac{1}{20} R_{pijq,kl}(p') + \frac{2}{45} R_{pijm}(p') R_{qklm}(p') \right) x^i x^j x^k x^l + O(r^5). \\ |g| &= 1 - \frac{1}{3} R_{ij} x^{ij} - \frac{1}{6} R_{ij,k}(p') x^{ijk} \\ &\quad - \left(\frac{1}{20} R_{ij,kl}(p') + \frac{1}{90} R_{hijm}(p') R_{hklm}(p') \right) x^i x^j x^k x^m + O(r^5). \end{aligned}$$

In the sequel, let us denote

$$x_{j_1 \cdots j_n}^{i_1 \cdots i_m} = x^{i_1 \cdots i_m j_1 \cdots j_n}, \quad \text{and} \quad \alpha_{j_1 \cdots j_n}^{i_1 \cdots i_m} = \frac{1}{2\pi^2} \int_{S^3} x^{i_1 \cdots i_m j_1 \cdots j_n} ds;$$

then around the point p' we write

$$\begin{aligned} g^{km} &= \delta^{km} + M^{km} = \delta^{km} + M_{km}^{ij} x^{km} + M_{kms}^{ij} x^{kms} + M_{kmst}^{ij} x^{kmst} + O(r^5) \\ M &= M^{ij} \delta_{ij} = M_{km} x^{km} + M_{kms} x^{kms} + M_{kmst} x^{kmst} + O(r^5), \\ \sqrt{|g|} &= 1 - \frac{1}{6} R_{ij} x^{ij} + K_{ijk} x^{ijk} + K_{ijkm} x^{ijkm} + O(r^5). \\ N^k &= -g^{ij} \Gamma_{ij}^k = N_i^k x^i + N_{ij}^k x^{ij} + N_{ijm}^k x^{ijm} + O(r^5). \end{aligned}$$

It is easy to check that $M_{km}^{ij} = -\frac{1}{3} R_{ikmj}(p')$, $M_{km} = \frac{1}{3} R_{ij}(p')$ and $N_i^k = -\frac{2}{3} R_{ik}(p')$.

We prove the following lemma.

Lemma 4.1. *We have*

$$\frac{1}{18}R_{ij}(p')R_{km}(p')\alpha^{ijkm} + N_{ijk}^m\alpha_m^{ijk} + M_{ijkm}\alpha^{ijkm} = 4K_{ijkm}\alpha^{ijkm}. \quad (4.1)$$

Proof. We have, for any small $t > 0$,

$$\begin{aligned} & \int_{B_t} \Delta_g r^2 dV_g \\ &= \int_{B_t} \left(8 - \frac{2}{3}R_{ij}x^{ij} + 2M_{ijk}x^{ijk} + 2M_{ijkm}x^{ijkm} + 2N_{ij}^k x_k^{ij} + 2N_{ijk}^p x_p^{ijk} \right) \\ & \quad \times \left(1 - \frac{1}{6}R_{ij}x^{ij} + K_{ijk}x^{ijk} + K_{ijkm}x^{ijkm} \right) dx + o(t^8) \\ &= 4\pi^2 t^4 - 2R_{ij}\alpha^{ij} \times 2\pi^2 \frac{t^6}{6} \\ & \quad + \left(\frac{1}{9}R_{ij}R_{km}\alpha^{ijkm} + 2M_{ijkm}\alpha^{ijkm} + 2N_{ijk}^p\alpha_p^{ijk} + 8K_{ijkm}\alpha^{ijkm} \right) 2\pi^2 \frac{t^8}{8} + o(t^8); \end{aligned}$$

on the other hand, we have

$$\begin{aligned} \int_{\partial B_t} 2r ds_g &= \int_{\partial B_t} 2r \left(1 - \frac{1}{6}R_{ij}x^{ij} + K_{ijkm}x^{ijkm} + O(r^5) \right) ds_0 \\ &= 4\pi^2 t^4 - 4\pi^2 \frac{R_{ij}}{6}\alpha^{ij} t^6 + 2K_{ijkm}\alpha^{ijkm} 2\pi^2 t^8 + o(t^8). \end{aligned}$$

Now the conclusion follows from Stokes' theorem. \square

Note that locally, we may write (see [Lemma A.1](#) in the [Appendix](#)),

$$G_{p'} = -2 \log r + S,$$

with

$$S = S_0(p') + a_i x^i + \frac{a_{ij}}{2} x^{ij} + O(r^{2+\alpha}).$$

We define

$$\varphi_\epsilon = -\log \left(1 + \lambda \left| \frac{x}{\epsilon} \right|^2 \right) + C_\epsilon + \mu |x|^2, \quad x \in B_{L\epsilon}$$

where

$$\mu = -\frac{1}{L^2 \epsilon^2 (1 + \lambda L^2)}, \quad \lambda = \frac{\sqrt{3\tilde{Q}(p')}}{12}$$

and

$$C_\epsilon = \log(1 + \lambda L^2) - 2 \log L\epsilon - \mu L^2 \epsilon^2.$$

We set

$$\phi_\epsilon = \begin{cases} G + \varphi_\epsilon + 2 \log r & x \in B_{L\epsilon} \\ G & x \notin B_{L\epsilon}, \end{cases}$$

then, in $B_{L\epsilon}$, we have

$$\phi_\epsilon = -\log\left(1 + \lambda \left|\frac{x}{\epsilon}\right|^2\right) + C_\epsilon + S + \mu|x|^2 = \varphi_\epsilon + S. \quad (4.2)$$

Hence, it is easy to check that $\phi_\epsilon \in W^{2,p}(M)$ for any $p > 0$.

We write

$$\begin{aligned} II(\phi_\epsilon) &:= \int_M \langle \phi_\epsilon, \phi_\epsilon \rangle dV_g + 4 \int_M Q_g \phi_\epsilon dV_g - 8\pi^2 \log \int_M \tilde{Q} e^{4\phi_\epsilon} dV_g \\ &= II_1 + II_2 + II_3. \end{aligned}$$

First we will calculate the term II_3 . In the small neighborhood around the point p' , we set

$$\tilde{Q} = \tilde{Q}(p') + b_{ij}x^i + \frac{b_{ij}}{2}x^{ij} + O(r^3),$$

then we have

$$\begin{aligned} \tilde{Q} e^{4\phi_\epsilon} \sqrt{|g|} &= \frac{e^{4C_\epsilon + 4S_0}}{\epsilon^4 \left(1 + \lambda \left|\frac{x}{\epsilon}\right|^2\right)^4} \left[(1 + 4a_i x^i + 2a_{ij} x^{ij} + 8a_i a_j x^{ij} + 4\mu r^2) \tilde{Q}(p') \right. \\ &\quad \left. + b_i x^i + \frac{b_{ij}}{2} x^{ij} + 4a_i b_i x^{ij} + O(r^{2+\alpha}) + O\left(\frac{r^2 \epsilon^2}{L^8}\right) \right] \\ &\quad \times \left(1 - \frac{R_{ij} x^{ij}}{6} + O(r^3) \right) \\ &= \frac{e^{4C_\epsilon + 4S_0}}{\epsilon^4 \left(1 + \lambda \left|\frac{x}{\epsilon}\right|^2\right)^4} \left[\left(1 + 4a_i x^i + 2a_{ij} x^{ij} + 8a_i a_j x^{ij} + 4\mu r^2 - \frac{R_{ij} x^{ij}}{6} \right) \right. \\ &\quad \left. \times \tilde{Q}(p') + b_i x^i + \frac{b_{ij}}{2} x^{ij} + 4a_i b_i x^{ij} + O(r^{2+\alpha}) + O\left(\frac{r^2}{L^8}\right) \right]. \end{aligned}$$

Therefore, by using the symmetry of the ball and the fact that $\alpha_{ij} = \frac{1}{4}\delta_{ij}$, we have

$$\begin{aligned} &\int_{B_{L\epsilon}} \tilde{Q} e^{4\phi_\epsilon} \sqrt{|g|} dV_g \\ &= 2\pi^2 e^{4C_\epsilon + 4S_0(p')} \epsilon^4 \int_0^L \frac{1}{(1 + \lambda r^2)^4} \left[\tilde{Q}(p') \left(1 + \epsilon^2 r^2 \left(\sum_i \left(\frac{a_{ii}}{2} + 2a_i^2 \right) \right. \right. \right. \\ &\quad \left. \left. \left. + 4\mu - \frac{R(p')}{24} \right) \right) + \sum_i \left(a_i b_i + \frac{b_{ii}}{8} \right) \epsilon^2 r^2 + O(\epsilon r)^{2+\alpha} + O\left(\frac{r^2}{L^4}\right) \right] r^3 dr. \end{aligned}$$

A direct calculation then yields that

$$\begin{aligned} 2\pi^2 \int_0^L \frac{r^3 dr}{(1 + \lambda r^2)^4} &= \frac{\pi^2}{6\lambda^2} + O\left(\frac{1}{L^4}\right), \\ 2\pi^2 \int_0^L \frac{r^5 dr}{(1 + \lambda r^2)^4} &= \frac{\pi^2}{3\lambda^3} + O\left(\frac{1}{L^2}\right), \end{aligned}$$

and

$$4\mu\epsilon^2 \times 2\pi^2 \int_0^L \frac{r^5 dr}{(1 + \lambda r^2)^4} = O\left(\frac{1}{L^4}\right).$$

Hence we get

$$\begin{aligned} & \int_{B_{L\epsilon}} \tilde{Q} e^{4\phi_\epsilon} \sqrt{|g|} dx \\ &= e^{4C_\epsilon + 4S_0} \epsilon^4 \left[8\pi^2 - \frac{24\pi^2}{\lambda^2 L^4} + \frac{\pi^2}{3\lambda^3} \epsilon^2 \left(\sum_i \left(\frac{a_{ii}}{2} + 2a_i^2 \right) \tilde{Q}(p') - \frac{R(p')}{24} \tilde{Q}(p') \right. \right. \\ & \quad \left. \left. + \sum_i \left(a_i b_i + \frac{b_{ii}}{8} \right) \right) \right] + O\left(\frac{1}{L^4}\right) + O(\epsilon^{2+\alpha}) + O\left(\frac{\epsilon^2}{L^2}\right). \end{aligned}$$

On the other hand, it is not difficult to check that

$$\begin{aligned} \int_{M \setminus B_{L\epsilon}} \tilde{Q} e^{4\phi_\epsilon} \sqrt{|g|} dx &= \int_{L\epsilon}^\delta \tilde{Q}(p') \frac{e^{4S_0}}{r^5} 2\pi^2 dr + O\left(\frac{1}{L^2 \epsilon^2}\right) \\ &= e^{4C_\epsilon + 4S_0} \epsilon^4 \left(\frac{24\pi^2}{\lambda^2 L^4} + O\left(\frac{\epsilon^2}{L^2}\right) \right). \end{aligned}$$

In conclusion, we have

$$\begin{aligned} & 8\pi^2 \log \int_M \tilde{Q} e^{4\phi_\epsilon} \sqrt{|g|} dx \\ &= 8\pi^2 [\log 8\pi^2 + 4(C_\epsilon + \log \epsilon + S_0)] \\ & \quad + \frac{\pi^2}{3\lambda^3} \left[\tilde{Q}(p') \sum_i \left(\frac{a_{ii}}{2} + 2a_i^2 \right) + \sum_i \left(a_i b_i + \frac{b_{ii}}{8} \right) - \frac{R(p')}{24} \tilde{Q}(p') \right] \epsilon^2 \\ & \quad + O(\epsilon^{2+\alpha}) + O\left(\frac{\epsilon^2}{L^2}\right) + O\left(\frac{1}{L^4}\right). \end{aligned} \tag{4.3}$$

Next, we calculate II_1 : first of all, by (4.2) we have

$$\begin{aligned} \int_M \langle \phi_\epsilon, \phi_\epsilon \rangle dV_g &= \int_{M \setminus B_{L\epsilon}} \langle \phi_\epsilon, \phi_\epsilon \rangle dV_g + \int_{B_{L\epsilon}} \langle \phi_\epsilon, \phi_\epsilon \rangle dV_g \\ &= \int_{M \setminus B_{L\epsilon}} \langle G, \phi_\epsilon \rangle dV_g + \int_{B_{L\epsilon}} \langle G, \phi_\epsilon \rangle dV_g \\ & \quad + \int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2 \log r, \phi_\epsilon \rangle dV_g \\ &= \int_M \langle G, \phi_\epsilon \rangle dV_g + \int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2 \log r, \phi_\epsilon \rangle dV_g \\ &= 16\pi^2 (C_\epsilon + S_0(p')) - 2 \int_M Q \phi_\epsilon dV_g \\ & \quad + \int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2 \log r, \varphi_\epsilon + S \rangle dV_g. \end{aligned} \tag{4.4}$$

We set η to be a cut-off function which is 0 at 1 and 1 in $[0, 1/4]$ with $\eta'(1) = 1$, and

$$h_\tau = \begin{cases} \eta\left(\frac{|x|}{\tau}\right) + \log \tau & |x| \leq \tau \\ \log r & |x| \geq \tau. \end{cases}$$

Then for fixed ϵ and L , we have

$$\lim_{\tau \rightarrow 0} \int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2h_\tau, \varphi_\epsilon + S \rangle dV_g = \int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2\log r, \varphi_\epsilon + S \rangle dV_g.$$

On the other hand, we have

$$\begin{aligned} & \int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2h_\tau, \varphi_\epsilon + S \rangle dV_g \\ &= \int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2h_\tau, G \rangle dV_g + \int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2h_\tau, \varphi_\epsilon + 2\log r \rangle dV_g \\ &= 16\pi^2 C_\epsilon + 32\pi^2 \eta(0) + 32\pi^2 \log \tau - 2 \int_{B_{L\epsilon}} Q_g(\varphi_\epsilon + 2h_\tau) \\ & \quad + \int_{B_{L\epsilon}} \langle \varphi_\epsilon, \varphi_\epsilon \rangle dV_g + \int_{B_{L\epsilon}} \langle \varphi_\epsilon, 2\log r + 2h_\tau \rangle dV_g + \int_{B_{L\epsilon}} \langle 2\log r, 2h_\tau \rangle dV_g. \end{aligned}$$

Therefore, we get

$$\begin{aligned} & \int_{B_{L\epsilon}} \langle \varphi_\epsilon + 2\log r, \varphi_\epsilon + S \rangle dV_g \\ &= 32\pi^2 \eta(0) - 2 \int_{B_{L\epsilon}} Q_g(\varphi_\epsilon + 2\log r) + \int_{B_{L\epsilon}} \langle \varphi_\epsilon, \varphi_\epsilon \rangle dV_g \\ & \quad + \int_{B_{L\epsilon}} \langle \varphi_\epsilon, 4\log r \rangle dV_g + \lim_{\tau \rightarrow 0} \left(\int_{B_{L\epsilon}} \langle 2\log r, 2h_\tau \rangle dV_g + 32\pi^2 \log \tau \right) \\ &= 32\pi^2 \eta(0) - 2 \int_{B_{L\epsilon}} Q_g(\varphi_\epsilon + 2\log r) + \int_{B_{L\epsilon}} \Delta_g \varphi_\epsilon \Delta_g \varphi_\epsilon dV_g \\ & \quad + 4 \int_{B_{L\epsilon}} \Delta_g \varphi_\epsilon \Delta_g \log r dV_g + \lim_{\tau \rightarrow 0} \left(\int_{B_{L\epsilon}} \Delta_g 2\log r \Delta_g 2h_\tau dV_g + 32\pi^2 \log \delta \right) \\ & \quad + \int_{B_{L\epsilon}} \frac{2}{3} R \langle d(\varphi_\epsilon + 2\log r), d(\varphi_\epsilon + 2\log r) \rangle dV_g \\ & \quad - \int_{B_{L\epsilon}} 2\text{Ric}(d(\varphi_\epsilon + 2\log r), d(\varphi_\epsilon + 2\log r)) dV_g. \end{aligned} \tag{4.5}$$

By a simple calculation, one gets

$$\begin{aligned} \int_{B_\tau} (\Delta_g 2\log r) \Delta_g (2h_\tau) dV_g &= \int_{B_\tau} \Delta_0(2\log r) \Delta_0 \left(2\eta \left(\frac{|x|}{\tau} \right) \right) dx + O(\tau) \\ &= -32\pi^2 \eta(0) + 16\pi^2 + O(\tau). \end{aligned} \tag{4.6}$$

To compute $\int_{B_{L\epsilon} \setminus B_\delta} \Delta_g \log r \Delta_g \log r$, we first verify that, for any function f which is smooth on $[t_0, t_1]$, where $t_0 < t_1$, we have

$$\begin{aligned} \Delta_g f(r) &= (\delta_{km} + M_{ij}^{km} x^{ij} + M_{ijs}^{km} x^{ijs} + M_{ijst}^{km} x^{ijst} + O(r^5)) \\ &\quad \times \left(f'' \frac{x_{km}}{r^2} + f' \frac{\delta_{km}}{r} - f' \frac{x_{km}}{r^3} \right) + N^k \frac{x_k}{r} f' \\ &= f'' + f' \left(\frac{3}{r} - \frac{R_{ij} x^{ij}}{3r} + \frac{M_{ijk} x^{ijk} + N_{ij}^k x_k^{ij}}{r} + \frac{M_{ijkm} x^{ijkm} + N_{ijk}^m x_m^{ijk}}{r} \right) \\ &\quad + O(r^5 |f''|) + O(r^4 |f'|). \end{aligned}$$

Here, we use the fact that $M_{ij}^{km} x_{km}^{ij} = M_{ijst}^{km} x_{km}^{ijst} = 0$. Then, applying Lemma 4.1, for any f_1 and f_2 which are smooth in $[t_0, t_1]$, we have

$$\begin{aligned} &\int_{B_{t_1} \setminus B_{t_0}} \Delta_g f_1(|x|) \Delta_g f_2(|x|) dV_g \\ &= \int_{t_0}^{t_1} f_1'' f_2'' \left(1 - \frac{R}{24} r^2 + K_{ijkm} \alpha^{ijkm} r^4 \right) 2\pi^2 r^3 dr \\ &\quad + \int_{t_0}^{t_1} (f_1' f_2'' + f_1'' f_2') \frac{1}{r} \left(3 - \frac{5R}{24} r^2 + 7K_{ijkm} \alpha^{ijkm} r^4 \right) 2\pi^2 r^3 dr \\ &\quad + \int_{t_0}^{t_1} f_1' f_2' \frac{1}{r^2} \left(9 + 33K_{ijkm} \alpha^{ijkm} r^4 - \frac{7R}{8} r^2 + \frac{1}{9} R_{ij} R_{km} \alpha^{ijkm} r^2 \right) 2\pi^2 r^3 dr \\ &\quad + \int_{t_0}^{t_1} \left(O(r^8 |f_1'' f_2''|) + O(r^7 (|f_1'' f_2'| + |f_1'| |f_2''|)) + O(r^6 |f_1' f_2'|) \right) \\ &= \int_{t_0}^{t_1} (f_1'' f_2'' + (f_1' f_2'' + f_1'' f_2') \frac{3}{r} + f_1' f_2' \frac{9}{r^2}) 2\pi^2 r^3 \\ &\quad + R \int_{t_0}^{t_1} \left(-f_1'' f_2'' \frac{r^2}{24} - \frac{5r}{24} (f_1' f_2'' + f_1'' f_2') - \frac{7}{8} f_1' f_2' \right) 2\pi^2 r^3 \\ &\quad + K_{ijkm} \alpha^{ijkm} \int_{t_0}^{t_1} (f_1'' f_2'' r^4 + 7(f_1' f_2'' + f_1'' f_2') r^3 + 33 f_1' f_2' r^2) 2\pi^2 r^3 dr \\ &\quad + R_{ij} R_{km} \alpha^{ijkm} \int_{t_0}^{t_1} \frac{1}{9} f_1' f_2' r^2 2\pi^2 r^3 dr \\ &\quad + \int_{t_0}^{t_1} \left(O(r^8 |f_1'' f_2''|) + O(r^7 (|f_1'' f_2'| + |f_1'| |f_2''|)) + O(r^6 |f_1' f_2'|) \right) dr. \end{aligned} \quad (4.7)$$

Then, choosing $f_1 = f_2 = 2 \log r$, $t_1 = L\epsilon$, $t_0 = \tau$, we get

$$\begin{aligned} \int_{B_{L\epsilon} \setminus B_\tau} \Delta_g(2 \log r) \Delta_g(2h_\tau) dV_g &= \int_{B_{L\epsilon} \setminus B_\tau} \Delta_g(2 \log r) \Delta_g(2 \log r) dV_g \\ &= 40K_{ijkm} \alpha^{ijkm} \pi^2 (L\epsilon)^4 + \frac{2\pi^2}{9} R_{ij} R_{km} \alpha^{ijkm} (L\epsilon)^4 \\ &\quad - 2R\pi^2 (L\epsilon)^2 + 32\pi^2 \log L\epsilon - 32\pi^2 \log \tau \\ &\quad + O(\tau) + O(L\epsilon)^5. \end{aligned} \quad (4.8)$$

Now we will calculate the term $\int_{B_{L\epsilon}} \Delta_g \varphi_\epsilon \Delta_g(\varphi_\epsilon + 4 \log r) dV_g$: in (4.7), we choose $f_1 = \varphi_\epsilon$, $f_2 = \varphi_\epsilon + 4 \log r$, $t_0 = 0$, $t_1 = L\epsilon$ then we get

$$\begin{aligned} \int_{B_{L\epsilon}} \Delta_g \varphi_\epsilon \Delta_g(\varphi_\epsilon + 4 \log r) dV_g &= -\frac{88}{3} \pi^2 + \frac{16\pi^2}{\lambda L^2} - 16\pi^2 \log(1 + \lambda L^2) \\ &\quad - R\epsilon^2 \frac{8\pi^2}{9\lambda} + 2\pi^2 R(L\epsilon)^2 - 40K_{ijkm} \alpha^{ijkm} \pi^2 (L\epsilon)^4 \\ &\quad - \frac{2\pi^2}{9} R_{ij} R_{km} \alpha^{ijkm} (L\epsilon)^4 + O(\epsilon^4 L^2) + \frac{\epsilon^2}{L^2} + O(L\epsilon)^5. \end{aligned} \quad (4.9)$$

By a direct calculation, we have

$$\begin{aligned} &\int_{B_{L\epsilon}} \frac{2}{3} R(\nabla_g(\varphi_\epsilon + 2 \log r), \nabla_g(\varphi_\epsilon + 2 \log r)) dV_g \\ &= \frac{2}{3} \int_0^{L\epsilon} R(p') \left(\frac{2\epsilon^2}{(\epsilon^2 + \lambda r^2)r} + 2\mu r \right)^2 2\pi^2 r^3 \\ &\quad + \frac{2}{3} \int_{B_{L\epsilon}} (R_{,i}(p') x^i + O(r^2)) \left(\frac{2\epsilon^2}{(\epsilon^2 + \lambda r^2)r} + 2\mu r \right)^2 (1 + O(r^3)) dx \\ &= \frac{8}{3\lambda} R(p') \pi^2 \epsilon^2 + \int_{B_{L\epsilon}} \left(\frac{2\epsilon^2}{(\epsilon^2 + \lambda r^2)r} + 2\mu r \right)^2 O(r^2) dx \\ &= \frac{8}{3\lambda} R(p') \pi^2 \epsilon^2 + O(\epsilon^4 L^2) + O\left(\frac{\epsilon^2}{L^2}\right), \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} &\int_{B_{L\epsilon}} 2\text{Ric}(\nabla_g(\varphi_\epsilon + 2 \log r), \nabla_g(\varphi_\epsilon + 2 \log r)) dV_g \\ &= \frac{1}{2} R(p') \int_0^{L\epsilon} \left(\frac{2\epsilon^2}{(\epsilon^2 + \lambda r^2)r} + 2\mu r \right)^2 2\pi^2 r^3 dr \\ &\quad + 2 \int_{B_{L\epsilon}} g^{is} g^{jt} (R_{ij,k}(p') x^k + O(r^2)) \left(\frac{2\epsilon^2}{(\epsilon^2 + \lambda r^2)r^2} + 2\mu \right)^2 x_{st} (1 + O(r^3)) dx \\ &= \frac{2}{\lambda} R(p') \pi^2 \epsilon^2 + 2 \int_{B_{L\epsilon}} (R_{ij,k}(p') x^k + O(r^2)) \\ &\quad \times \left(\frac{2\epsilon^2}{(\epsilon^2 + \lambda r^2)r^2} + 2\mu \right)^2 x^{ij} (1 + O(r^3)) dx \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{\lambda} R(p') \pi^2 \epsilon^2 + \int_{B_{L\epsilon}} \left(\frac{2\epsilon^2}{(\epsilon^2 + \lambda r^2)r^2} + 2\mu \right)^2 O(r^4) dx \\
&= \frac{2}{\lambda} R(p') \pi^2 \epsilon^2 + O(\epsilon^4 L^2) + O\left(\frac{\epsilon^2}{L^2}\right).
\end{aligned} \tag{4.11}$$

Together with (4.4)–(4.6) and (4.8)–(4.11), we obtain the following identity

$$\begin{aligned}
II_\epsilon(u_\epsilon) &= II_1 + II_2 + II_3 \\
&= -16\pi^2 \log \lambda - 8\pi^2 \log 8\pi^2 + \frac{8\pi^2}{3} - 16\pi^2 + 2 \int_M QG - 16\pi^2 S_0 \\
&\quad - \frac{\epsilon^2 \pi^2}{3\lambda^3} \left(\tilde{Q}(p') \sum_i \left(\frac{a_{ii}}{2} + 2a_i^2 \right) + \sum_i \left(a_i b_i + \frac{b_{ii}}{8} \right) - \frac{R(p')}{36} \tilde{Q}(p') \right) \\
&\quad + O\left(\frac{\epsilon^2}{L^2}\right) + O(\epsilon^{2+\alpha}) + O\left(\frac{1}{L^4}\right) + O(\epsilon^4 L^2) + O((L\epsilon)^5).
\end{aligned} \tag{4.12}$$

Proof of Theorem 1.2. We set $L = \frac{\log \frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$, then

$$\epsilon^2 \gg O\left(\frac{\epsilon^2}{L^2}\right) + O(\epsilon^{2+\alpha}) + O\left(\frac{1}{L^4}\right) + O(\epsilon^4 L^2) + O((L\epsilon)^5)$$

when ϵ is very small. Therefore, we get Theorem 1.2. \square

5. The local conformally case

In this section, we will discuss the local conformally flat case of Theorem 1.2.

In this situation, locally we may write

$$g = e^{2f} \sum_i dx^i \otimes dx^i \quad \text{with } f = c_i x^i + \frac{1}{2} c_{ij} x^{ij} + O(r^3),$$

and

$$\tilde{Q} = \tilde{Q}(p') + b_i x^i + \frac{1}{2} b_{ij} x^{ij} + O(r^3).$$

Note that by the conformal property of P_g , the corresponding Green function has the following local expression:

$$G = -2 \log |x| + S_0(p') + a_i x^i + \frac{1}{2} a_{ij} x^{ij} + O(r^3).$$

When $f = 0$, we can use Theorem 1.2 to obtain: if

$$\sum_i \left(\frac{a_{ii}}{2} + 2a_i^2 + \frac{1}{\tilde{Q}(p')} \left(a_i b_i + \frac{b_{ii}}{8} \right) \right) > 0,$$

then (1.3) has a solution.

For the general case, we set $g' = e^{-2f}g$, then applying Lemma 3.6, we get $G'_{p'} = G + f$, and then

$$a'_i = a_i + c_i, \quad \text{and} \quad a'_{ii} = a_{ii} + c_{ii}.$$

Thus we have the following results.

Theorem 5.1. *Let (M, g) be a closed 4-dimensional manifold with $k = 8\pi^2$ and let P_g be positive. Suppose further that it is locally conformal flat near p' . If*

$$\sum_i \frac{a_{ii} + c_{ii}}{2} + 2(a_i + c_i)^2 + \frac{1}{\tilde{Q}(p')} \left((a_i + c_i)b_i + \frac{b_{ii}}{8} \right) > 0,$$

then Eq. (1.3) has a minimal solution.

As a corollary, we have the following.

Corollary 5.2. *With the same assumption as in Theorem 5.1. If*

$$\sum_i \frac{a_{ii} + c_{ii}}{2} + 2(a_i + c_i)^2 > 0,$$

then in the conformal class of (M, g) there is a constant Q -curvature.

To end this section, we propose the following conjecture.

Conjecture. *Let (M, g) be a locally conformal flat closed Riemannian manifold of dimension four, with $k = 8\pi^2$ and let P_g be positive. Then we have*

$$\sum_i \left(\frac{a_{ii} + c_{ii}}{2} + 2(a_i + c_i)^2 \right) \geq 0, \quad \text{at the point } p' \text{ where } \Lambda_g(p') = \min_{x \in M} \Lambda_g(8\pi^2, x),$$

and the equality holds if and only if (M, g) is in the conformal class of the standard 4-sphere.

Let $\tilde{g} = e^{2G}g$; then we have

$$Q_{\tilde{g}}(x) = 0$$

for any $x \neq p$. Near p , we can write

$$\tilde{g} = \frac{e^{S_0(p) + (c_i + a_i)x^i + (c_{ij} + a_{ij})x^{ij}}}{r^2} = \frac{e^{S_0(p)}}{r^2} (\theta_i x^i + \theta_{ij} x^{ij} + O(|x|^3)).$$

So the above conjecture is equivalent to

$$\sum_i \theta_{ii} > 0$$

when $M \neq S^4$. So, this problem is very similar to the positive mass problem.

Acknowledgments

The authors thank the referee for his helpful comments. The research was supported by the National Natural Science Foundation of China, Nos11071236 and 11131007.

Appendix

Suppose $\text{Ker } P_g = \{\text{constant}\}$. Let G be the Green function which satisfies

$$P_g G + 2Q_g = 16\pi^2 \delta_p.$$

As a corollary of a result in [17], we have the following.

Lemma A.1. *In a normal coordinate system of p , we have*

$$G = -2 \log r + S_0 + a_i x^i + a_{ij} x^{ij} + O(r^{2+\alpha}).$$

However, for the reader's sake, we give a brief proof of this lemma here.

Proof. In a normal coordinate system, we set

$$|g| = 1 - \frac{1}{3} R_{ij} x^{ij} + O(r^3), \quad \text{and} \quad g^{km} = \delta^{km} - \frac{1}{3} R_{kijm} x^{ij} + O(r^3)$$

where φ_{ijk} and θ_{ijk} are smooth.

Given a smooth function F , we have

$$\begin{aligned} \Delta_g F(|x|) &= \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^k} \left(\sqrt{|g|} g^{km} \frac{\partial}{\partial x^m} F \right) \\ &= \frac{\partial}{\partial x_k} \left(g^{km} F' \frac{x_m}{r} \right) + \frac{1}{2} g^{km} F_m \frac{\partial}{\partial x_k} \log |g| \\ &= \frac{\partial}{\partial x_k} \left(F' \frac{x_k}{r} - \frac{1}{3} R_{kijm} F' \frac{x^{kij}}{r} + F' O(r^3) \right) - \frac{1}{3} R_{ij} F' \frac{x^{ij}}{r} + O(F' r^2) \\ &= \frac{\partial}{\partial x_k} \left(F' \frac{x_k}{r} + F' O(r^3) \right) - \frac{1}{3} R_{ij} F' \frac{x^{ij}}{r} + O(F' r^2) \\ &= \Delta_0 F - \frac{1}{3} R_{ij} F' \frac{x^{ij}}{r} + O(F' r^2) + O(F'' r^3). \end{aligned}$$

Then

$$\Delta_g (-2 \log r) = -\frac{4}{r^2} + \frac{2}{3} R_{ij} \frac{x^{ij}}{r^2} + O(r)$$

and

$$\Delta_g \left(-\frac{4}{r^2} \right) = \Delta_0 \left(-\frac{4}{r^2} \right) - \frac{8 R_{ij} x^{ij}}{3r^4} + O\left(\frac{1}{r}\right) = 16\pi^2 \delta_0 - \frac{8 R_{ij} x^{ij}}{3r^4} + O\left(\frac{1}{r}\right).$$

It is easy to check that

$$\Delta_g \frac{2}{3} R_{ij} \frac{x^{ij}}{r^2} = \Delta_0 \frac{2}{3} R_{ij} \frac{x^{ij}}{r^2} + O\left(\frac{1}{r}\right) = \frac{4R}{3r^2} - \frac{16 R_{ij} x^{ij}}{3r^4}.$$

Hence, we get

$$\Delta_g^2 (-2 \log r) = 16\pi^2 \delta_p + \frac{4R}{3r^2} - 8 \frac{R_{ij} x^{ij}}{r^4} + O\left(\frac{1}{r}\right).$$

Moreover, we have

$$\begin{aligned} & \operatorname{div} \left(\frac{2}{3} R_g(-2 \log r) - 2 \operatorname{Ric}_g \langle d(-2 \log r), \cdot \rangle \right) \\ &= \frac{2}{3} R_p(p')(2 \log r)_{kk} - 2 R_{km}(p')(2 \log r)_{km} + O\left(\frac{1}{r}\right) \\ &= \frac{2}{3} R_g(p') \frac{4}{r^2} - 4 R_g(p') \frac{1}{r^2} + 8 R_{km} \frac{x^{km}}{r^4} + O\left(\frac{1}{r}\right). \end{aligned}$$

We therefore have

$$P_g(-2 \log r) = 16\pi^2 \delta_0 + O\left(\frac{1}{r}\right).$$

We set

$$G = -2 \log r + S$$

where $S \in C^{1,\alpha}$. Then, we get

$$\Delta_g^2 S = P_g S + O\left(\frac{1}{r}\right) = P_g G + 2P_g \log r + O\left(\frac{1}{r}\right) = O\left(\frac{1}{r}\right).$$

This proves the lemma. \square

References

- [1] F. Adimurthi, M. Robert, Struwe: concentration phenomena for Liouville's equation in dimension four, *J. Eur. Math. Soc.* 8 (2006) 171–180.
- [2] S. Brendle, Convergence of the Q -curvature flow on S^4 , *Adv. Math.* 205 (2006) 1–32.
- [3] S.-Y.A. Chang, P.C. Yang, Extremal metrics of zeta functional determinants on 4-manifolds, *Ann. of Math.* 142 (1995) 172–212.
- [4] X. Chen, X. Xu, The scalar curvature flow on S^n -perturbation theorem revisited, *Invent. Math.* 187 (2012) 395–506.
- [5] X. Chen, X. Xu, Q -curvature flow on the standard sphere of even dimension, *J. Funct. Anal.* 261 (2011) 934–980.
- [6] W.Y. Ding, J. Jost, J. Li, G. Wang, The differential equation $\Delta u = 8\pi - 8\pi h e^u$ on a compact Riemann surface, *Asian J. Math.* 1 (1997) 230–248.
- [7] Z. Djadli, A. Malchiodi, Existence of conformal metrics with constant Q -curvature, *Ann. of Math.* 168 (2008) 813–858.
- [8] O. Druet, F. Robert, Bubbling phenomena for fourth-order four-dimensional PDEs with exponential growth, *Proc. Amer. Math. Soc.* 134 (2006) 897–908.
- [9] M. Gursky, The principle eigenvalue of a conformally invariant differential operator, with an application to semilinear elliptic PDE, 1998. Preprint.
- [10] J.M. Lee, T.H. Parker, The Yamabe problem, *Bull. Amer. Math. Soc. (NS)* 17 (1987) 37–91.
- [11] Y. Li, Moser–Trudinger inequality on a compact Riemannian manifold of dimension two, *J. Partial Differential Equations* 14 (2001) 163–192.
- [12] J. Li, Y. Li, Solutions for Toda systems on Riemann surfaces, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* (2005) 703–728.
- [13] C.S. Lin, A classification of solutions of conformally invariant fourth order equation in \mathbb{R}^n , *Comment. Math. Helv.* 73 (1998) 206–231.
- [14] A. Malchiodi, Compactness of solutions to some geometric fourth-order equations, *J. Reine Angew. Math.* 594 (2006) 137–174.
- [15] A. Malchiodi, Conformal metrics with constant Q -curvature, *SIGMA Symmetry Integrability Geom. Methods Appl.* 3 (2007) 11. Paper 120.
- [16] A. Malchiodi, M. Struwe, Q -curvature flow on S^4 , *J. Differential Geom.* 73 (2006) 1–44.
- [17] C.B. Ndiaye, Constant Q -curvature metrics in arbitrary dimension, *J. Funct. Anal.* 251 (2007) 1–58.
- [18] S. Paneitz, A quartic conformally covariant differential operator for pseudo-Riemannian manifolds, 1983. Preprint.

- [19] J. Qing, D. Raske, Compactness for conformal metrics with constant Q curvature on locally conformally flat manifolds, *Calc. Var. Partial Differential Equations* 26 (2006) 343–356.
- [20] J. Wei, X. Xu, On conformal deformations of metrics on S^n , *J. Funct. Anal.* 157 (1998) 292–325.
- [21] G. Weinstein, L. Zhang, The profile of bubbling solutions of a class of fourth order geometric equations on 4-manifolds, *J. Funct. Anal.* 257 (2009) 3895–3929.